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CONTINUOUS FUNCTIONS ON COUNTABLE COMPACT ORDERED SETS AS SUMS OF THEIR INCREMENTS

BY

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ABSTRACT. Every continuous function from a countable compact linearly ordered set A into a Banach space V (vanishing at the least element of A) admits a representation as a sum of a series of its increments (in the topology of uniform convergence). This series converges to no other sum under rearrangements of its terms. A uniqueness result to the problem of representation of a regulated real function on the unit interval as a sum of a continuous and a steplike function is derived.

0. Introduction. Let A be a compact linearly ordered set, V a (real or complex) Banach space. Let $C(A, V)$ denote the Banach space of continuous functions from A to V with the supremum norm. The right neighbour a' (the left neighbour $'a$) of $a \in A$ is defined as the maximal (minimal) $b \in A$ such that $(a, b) = \varnothing$ ($(b, a) = \varnothing$). With $f: A \to V$ associate $f^{\dagger}: A \to V$ defined by

$$f^{\dagger}(a) = f(a') - f(a).$$

With $a \in A$, $v \in V$ associate a function $J_a^v: A \to V$—*the v-jump at a*—defined by

$$J_a^v(b) = \begin{cases} 0, & b \leq a, \\ v, & a < b. \end{cases}$$

Then $J_a^{f^{\dagger}(a)}$ is called *the increment of f at a*. Note that if f is in $C(A, V)$ so is every increment of f. Let m_A (M_A) denote the minimal (maximal) element of A. The purpose of this article is to prove

THEOREM 1. *Let A be a countable compact ordered set, V a Banach space and let $f \in C(A, V)$ satisfy $f(m_A) = 0$. Then*:

(a) *There is an enumeration $(J_n)_n$ of the increments of f such that $f = \sum_n J_n$ holds in $C(A, V)$.*

(b) *If $(\bar{J}_n)_n$ is any enumeration of the increments of f for which $\sum_n \bar{J}_n$ converges in $C(A, V)$, then $\sum_n \bar{J}_n = f$.*

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In the special case where $\nu = \rho + 1$ is a compact countable ordinal Theorem 1(a) implies: Let f: $\nu \to V$ be continuous. Then there is an enumeration $(\mu_n)_n$ of the ordinals smaller than ν such that for every $\mu < \nu$, $f(\mu) = f(0) + \Sigma_{\mu_n < \mu}(f(\mu_n + 1) - f(\mu_n))$. Moreover, the convergence is uniform in μ (in a sense made precise in §§2, 3). Similarly, there is an enumeration $\bar{\mu}_n$ of the ordinals smaller than ν so that the identity $f(\mu) = f(\rho) + \Sigma_{\mu < \bar{\mu}_n}(f(\bar{\mu}_n) - f(\bar{\mu}_n + 1))$ holds uniformly for $\mu < \nu$.

Conditionally convergent series, every convergent rearrangement of which have the same sum, are known to exist in any infinite dimensional Banach space **[Ha]**, **[McA]**. Theorem 1 provides a host of natural examples of this phenomenon. The simplest of those is the following one: Let $A = \omega + 1 = \{0, 1, 2, \ldots, \omega\}$ and let $V = R$, the reals. Let $(a_n)_n$ be any sequence of reals such that $\Sigma_n a_n$ is conditionally convergent, and $\Sigma_n a_n = a$. Define $f \in C(A, R)$ by $f(n) = \Sigma_{i<n} a_i$, $f(\omega) = a$. It is straightforward to check that $(J_n^{a_n})_n$ is an enumeration of f's increments, $\Sigma_n J_n^{a_n} = f$, and that $\Sigma_n J_n^{a_n}$ is convergent in $C(A, R)$ iff $\Sigma_n a_n$ is convergent to a. Thus, $\Sigma_n J_n^{a_n}$ is a conditionally convergent series in $C(A, R)$, every convergent rearrangement of which has the sum f.

Although V is assumed to be Banach space in Theorem 1, the theorem is true in a wider context. We may assume V to be arbitrary complete normed abelian group.[1] (A typical example of such a V which is not the additive group of a Banach space is the p-adic ring Q_p, see e.g. **[F]**.)

Let A be arbitrary ordered set. Call $a \in A$ *right isolated* (*left isolated*) iff $a < a'$ $('a < a)$. Call a *a core point* iff it is neither right nor left isolated. Let f: $A \to V$. We write $f(a+) = v$ iff $\forall \varepsilon > 0\ \exists b > a\ [a < c < b \Rightarrow \|f(c) - v\| < \varepsilon]$. Define $f(a-)$ similarly. Note that $f(a) = f(a+)$ $(f(a) = f(a-))$ whenever a is right (left) isolated. Let $f(m_A -) = f(m_A)$, $f(M_A +) = f(M_A)$. We call f: $A \to V$ a *regulated function* iff:

(i) $f(m_A +)$, $f(M_A -)$ exist, as do $f(a+)$, $f(a-)$ for $m_A < a < M_A$.

(ii) $f(a) = f(a-)$ whenever a is right isolated,

$f(a) = f(a+)$ whenever a is left isolated,

$f(a) = f(a-)$ whenever a is a core point.

The reader will note that a regulated function is continuous if and only if it satisfies $f(a-) = f(a+)$ for every core point a.

If A is compact, the family of all regulated functions from A to V with the supremum norm is again a Banach space $\mathrm{Reg}(A, V)$ containing $C(A, V)$. For $f \in \mathrm{Reg}(A, V)$ define f^*: $A \to V$ by $f^*(a) = f(a+) - f(a)$ if a is a core point, and $f^*(a) = f^\dagger(a)$ otherwise. *The increment of f at a* is redefined to be $J_a^{f^*(a)}$. Theorem 1 is equivalent to

[1] By a norm on an additive group V we mean a function $\| \ \|$ from V into the nonnegative real numbers, satisfying $\|v\| = 0$ iff $v = 0$, $\|v\| = \|-v\|$ and $\|v + \omega\| \le \|v\| + \|\omega\|$.

THEOREM 2. *Let A be a compact countable ordered set, V a Banach space, and let $f \in \mathrm{Reg}(A, V)$ satisfy $f(m_A) = 0$. Then*:

(a) *There is an enumeration $(J_n)_n$ of the increments of f such that $f = \sum_n J_n$ holds in* $\mathrm{Reg}(A, V)$.

(b) *If $(\bar{J}_n)_n$ is any enumeration of the increments of f for which $\sum_n \bar{J}_n$ converges in* $\mathrm{Reg}(A, V)$, *then $\sum_n \bar{J}_n = f$.*

Indeed, $f^* = f^\dagger$ for $f \in C(A, V)$ and so Theorem 2 clearly implies Theorem 1. Conversely, assume Theorem 1 and let A be a countable compact ordered set. Replacing each core point $a \in A$ by a pair of points $\bar{a} < \overset{+}{a}$, we obtain another compact countable ordered set $\hat{A}$ with no core points. Define $g\colon \hat{A} \to A$ by $g(\bar{a}) = g(\overset{+}{a}) = a$ for a core point $a \in A$, $g(a) = a$ for a noncore point $a \in A$. For $f \in \mathrm{Reg}(A, V)$ let $\hat{f} = f \circ g$. Then $\hat{f} \in C(\hat{A}, V)$ and it is easily checked that Theorem 1 for $\hat{f}$ implies Theorem 2 for f (compare Proposition 3.1).

We apply Theorem 2 to obtain a theorem on $\mathrm{Reg}(I, V)$, where $I = [0, 1]$ is the closed unit interval (see [GMW], [M]). $s \in \mathrm{Reg}(I, V)$ is called *steplike* iff there is an enumeration $(J_n)_n$ of its increments (only countably many of which are nonzero) so that $s = \sum_n J_n$ holds in $\mathrm{Reg}(I, V)$. The separation of discontinuities of an $f \in \mathrm{Reg}(I, V)$ by means of a steplike function dates back essentially to Lebesgue's Theorem on monotone real functions: every monotone real function has a unique representation $f = g + s$, where g is a monotone continuous function and s is a monotone steplike function. Similarly, every $f \in \mathrm{Reg}(I, V)$ of bounded variation has a unique representation $f = g + s$, where $g \in C(I, V)$ and s is steplike. In general, however, such a representation need not exist, and when it exists need not be unique [GMW], [M]. Call $f \in \mathrm{Reg}(I, V)$ *representable* if f has a representation $f = g + s$, where $g \in C(I, V)$ and $s \in \mathrm{Reg}(I, V)$ is steplike, and *uniquely representable* if f has precisely one such representation. Obviously, the representability and unique representabiliy of $f \in \mathrm{Reg}(I, V)$ depend only on f^*. Now $f \to f^*$ is a continuous linear mapping of $\mathrm{Reg}(I, V)$ onto $C_0(I, V) = \{h\colon I \to V \mid \{t\colon \|h(t)\| > \varepsilon\}$ is finite for every $\varepsilon > 0\}$ with the supremum norm. Call $h \in C_0(I, V)$ *summable* (*uniquely summable*) iff some–or equivalently, any–f in $\mathrm{Reg}(I, V)$ with $f^* = h$ is representable (uniquely representable).

The complete characterization of summable or uniquely summable members of $C_0(I, R)$ is still open. The methods of this paper suffice to characterize, however, those summable members of $C_0(I, R)$ whose support has a countable closure (to appear elsewhere). We shall address ourselves here only to the uniqueness problem.

In [M] it is shown that an $h \in C_0(I, R)$ exists, such that every $f \in \mathrm{Reg}(I, R)$ with $f(0) = 0$ and $f^* = h$ is steplike. Obviously, the support of such an h is necessarily dense in I, and so has I for its closure. In a similar way, it can be shown that given any countable set A in I of uncountable closure, there is an $h \in C_0(I, R)$ whose support is A such that the family $\{s \in \operatorname{Reg}(I, R)$: s is steplike and $s^* = h\}$ is of the cardinality of the continuum. This is not anymore possible if h's support is of countable closure. The following fact was stated in [M] for the case $V = R$:

THEOREM 3. *Let A be a countable closed subset of I. Let $f \in \operatorname{Reg}(I, V)$ satisfy $\{t: f^*(t) \neq 0\} \subseteq A$. If f is representable, then f is uniquely representable.*

PROOF. Let $W \subseteq \operatorname{Reg}(I, V)$ be the closed subspace of those $f \in \operatorname{Reg}(I, V)$ that vanish at 0 and are constant on every component of $I - A$. For $f: I \to V$ let $f_A: A \to V$ denote the restriction of f to A. Clearly, $f_A \in \operatorname{Reg}(A, V)$ whenever $f \in W$, and the mapping $Tf = f_A$ is a linear isometry of W onto $\operatorname{Reg}_0(A, V) = \{f \in \operatorname{Reg}(A, V): f(m_A) = 0\}$.

We show now that $W = \{s \in \operatorname{Reg}(I, V)$: s is steplike and $\{t: s^*(t) \neq 0\} \subseteq A\}$. Clearly $s \in W$ whenever s is steplike and $\{t: s^*(t) \neq 0\} \subseteq A$. Conversely, let $f \in W$. By Theorem 2(a), there is an enumeration $(J_n)_n$ of f_A's increments so that $f_A = \Sigma_n J_n$. Apply T^{-1} and obtain $f = \Sigma_n T^{-1}J_n$ in $\operatorname{Reg}(I, V)$. But $(T^{-1}J_n)_n$ is obviously an enumeration of f's increments, and so f is steplike.

Now assume that $f \in \operatorname{Reg}(I, V)$ is representable, and that $\{t: f^*(t) \neq 0\} \subseteq A$. Let $f = g_1 + s_1 = g_2 + s_2$ where $g_1, g_2 \in C(I, V)$ and s_1, s_2 steplike. By $f^* = s_1^* = s_2^*$ we have $s_1, s_2 \in W$, and $(Ts_1)^* = (Ts_2)^* = f_A^*$. Hence $Ts_1, Ts_2 \in \operatorname{Reg}(A, V)$ have the same increments and vanish at m_A. By Theorem 2(b), $Ts_1 = Ts_2$, whence $s_1 = s_2$ and $g_1 = g_2$. □

Theorem 1 is proved essentially by induction on the Cantor rank of the scattered space A. In §1 the class of countable compact order types is characterized as the smallest class of order types including $\bar{0}$ and closed under one infinitary operation, the compact-limit operation (Definition 1.1). This yields a useful induction principle for countable compact order types. §2 is devoted to the proof of a rather technical summing lemma, which is the core of the inductive proof of Theorem 1, given in §3.

We are indebted to Casper Goffman for many hours of critical, useful and enjoyable discussions.

1. An induction principle for countable compact order types. We specify some notation first. An ordinal number is identified with the set of smaller ordinals and so $\mu < \nu$ and $\mu \in \nu$ are interchangeable. A cardinal number is an ordinal not equivalent to a smaller ordinal. ω denotes the first infinite cardinal. $|A|$ denotes the cardinality of A, that is, the cardinal number equivalent to the set A. An *order* on A (or an *ordering of* A) is a total irreflexive and transitive relation on A. Let $<$ be an order on A, B, $C \subseteq A$, then $B < C$ means $b < c$ for all $b \in B$, $c \in C$. Note that $\varnothing < B < \varnothing$ holds for every $B \subseteq A$. $B < x$ ($x < B$) stands for $B < \{x\}$ ($\{x\} < B$). The order type of A is denoted by $\bar{A}$. The reader is referred to [E] for the definitions of an order type and of arithmetical operations on ordered sets and order types. Intervals in an ordered set are denoted in the usual manner, e.g. $[a, b) = \{c \in A\colon a \leq c < b\}$, $(a, b) = \{c \in A\colon a < c < b\}$. Ordinals are considered as ordered sets, being well ordered by the membership relation. An ordered set is considered as a topological space, with the topology generated by the sets $\{b\colon b < a\}$ and $\{b\colon a < b\}$. The following proposition will be useful ([HK], see e.g. [Ke, p. 162]):

Proposition 1.0. *Let A be a nonempty ordered set. Then A is compact if and only if:*

(i) *A has a smallest element,*

(ii) *every nonempty subset of A has a least upper bound in A.*

Definition 1.1. Let A_n be an ordered set and let $\alpha_n = \bar{A}_n$ ($n \in \omega$). Assume further that $A_n \cap A_m = \varnothing$ for $n \neq m$, $x \notin \bigcup_{n\in\omega} A_n$ and denote by $<_n$ the ordering of A_n. Define the *x-compact-limit-sum of* $(A_n)_{n\in\omega}$, denoted x-$\mathrm{CL}_n A_n$, as follows. The domain of x-$\mathrm{CL}_n A_n$ is $\{x\} \cup \bigcup_{n\in\omega} A_n$. The ordering on x-$\mathrm{CL}\, A_n$ is defined by:

(1) $a < b$ iff $a <_n b$ for $a, b \in A_n$.

(2) $A_{2n} < A_{2n+2} < x < A_{2n+3} < A_{2n+1}$, $n \in \omega$.

Let α be the order type of x-$\mathrm{CL}_{n\in\omega} A_n$. Then α is defined as the *compact-limit-sum of* $(\alpha_n)_{n\in\omega}$, denoted by $\alpha = \mathrm{CL}_n\, \alpha_n$. The rest of this section is devoted to proving

Theorem 1.2. *Let $\mathcal{C}$ denote the class of countable compact order types. Then $\mathcal{C}$ is the smallest class of order types including $\bar{0}$ and closed under the compact-limit-sum operation.*

As a corollary we have the following

Induction Principle. Let $P(A)$ be a statement on ordered sets. *If* $P(\varnothing)$ is true, and $P(x\text{-}\mathrm{CL}_{n\in\omega} A_n)$ is true whenever $P(A_n)$ is true for every $n \in \omega$, *then* $P(A)$ is true for every countable compact ordered set.

Let $\mathcal{C}_0$ denote the smallest class of order types including $\bar{0}$ and closed under the compact-limit-sum operation. Since x-$\mathrm{CL}_n A_n$ is countable whenever each A_n is, and by Proposition 1.0 it is compact whenever each A_n is, we have $\mathcal{C}_0 \subseteq \mathcal{C}$.

The reverse inclusion is proved essentially by induction or the Cantor rank $\mathrm{rk}(\alpha)$ of α. We first resume some notation and facts (see e.g. [Ke], [Ku]). For a topological space, X, X' denotes the set of nonisolated points of X. Define by induction $X^{(\mu)}$ for every ordinal μ as follows. $X^{(0)} = X$, $X^{(\mu+1)} = X^{(\mu)\prime}$, and $X^{(\nu)} = \bigcap_{\mu<\nu} X^{(\mu)}$ when ν is a limit ordinal. Then $X^{(\nu)} \subseteq X^{(\mu)}$ for $\mu < \nu$, and $\mathrm{rk}(X)$ is defined as the least ordinal μ for which $X^{(\mu)} = X^{(\mu+1)}$. X is called *scattered* iff $X^{(\nu)} = \varnothing$ for $\nu \geq \mathrm{rk}(X)$.

Let X be a scattered compact nonempty Hausdorff space. Then $\mathrm{rk}(X) = \mu + 1$ for some ordinal μ, and $X^{(\mu)}$ is finite. Define $\mathrm{ch}(X) = (\mu, m)$ where $\mathrm{rk}(X) = \mu + 1$ and $|X^{(\mu)}| = m > 0$. Also let $\mathrm{ch}(\varnothing) = (0, 0)$.

Every countable compact set of reals X is scattered, its induced topology coincides with its order topology, and its rank is countable. Every countable compact ordered set is order isomorphic with a compact countable set of reals, and so $\mathcal{C}$ is actually the class of order types of countable compact sets of reals.

We define $\mathrm{ch}(\alpha)$ for a compact order type α by $\mathrm{ch}(\alpha) = \mathrm{ch}(A)$, where A is an ordered set of type α. In view of the previous remarks the following is clear.

Proposition 1.3. *Let* $\alpha \in \mathcal{C}$, $\mathrm{ch}(\alpha) = (\mu, m)$. *Then there are* α_i, $i < m$ *such that* $\alpha = \sum_{i<m} \alpha_i$ *and* $\mathrm{ch}(\alpha_i) = (\mu, 1)$.

Proposition 1.4. *Let A be a scattered ordered set and let x be a nonisolated point of A. Then x is an accumulation point of isolated points. Moreover, if $m_A < x < M_A$ is a core point, then for every $a, b \in A$ with $a < x < b$ there are isolated points $\bar{a}, \bar{b} \in A$ such that $a < \bar{a} < x < \bar{b} < b$.*

Proof. Let $\mu(x)$ be the ordinal μ satisfying $x \in A^{(\mu)} - A^{(\mu+1)}$. Then x is nonisolated iff $\mu(x) > 0$. Also, if $\mu(x) > 0$ then there is an interval around x containing in addition only y's with $\mu(y) < \mu(x)$. The proposition follows by a straightforward induction. $\square$

Corollary 1.5. *Let A be a countable scattered ordered set and let x be a core point in A satisfying $m_A < x < M_A$. Then there are sequences $(a_n)_{n\in\omega}$ and $(b_n)_{n\in\omega}$ of isolated points in A such that $a_n < a_{n+1} < x < b_{n+1} < b_n$, and $x = \sup_n a_n = \inf_n b_n$.*

Order the class $\{(\mu, m)\colon \mu \text{ an ordinal}, m \in \omega\}$ lexicographically: $(\mu, m) < (\tilde{\mu}, \tilde{m})$ if $\mu < \tilde{\mu}$ or $\mu = \tilde{\mu}$ and $m < \tilde{m}$. A proof by induction on $\mathrm{ch}(A)$ or $\mathrm{ch}(\alpha)$ will refer to this well-ordering. We prove now $\mathcal{C} \subseteq \mathcal{C}_0$. Let $\alpha \in \mathcal{C}$, let $\mathrm{ch}(\alpha) = (\mu, m)$ and proceed by induction. Let $\mu = 0$ and proceed by induction on m. Since $\bar{0} \in \mathcal{C}_0$ the case $m = 0$ is done. Now $\mathrm{ch}(\alpha) = (0, m)$ iff $\alpha = \bar{m}$. Assume $\bar{m} \in \mathcal{C}_0$. Let $\alpha_0 = \bar{m}$, $\alpha_n = \bar{0}$ for $n > 0$. Then $\mathrm{CL}_n\, \alpha_n = \overline{m+1}$, and so $\overline{m+1} \in \mathcal{C}_0$.

Assume now $\mu > 0$. Let first $m = 1$ and let A be an ordered set of of order type α. Let $A^{(\mu)} = \{x\}$. By Corollary 1.5 pick a monotone sequence $(a_n)_{n\in\omega}$ of isolated points converging to x, say $a_n < a_{n+1} < x$. Define A_{2n} by $A_0 = [m_A, a_0]$, $A_{2n+2} = [a_n', a_n]$. Since a_n is isolated and $a_n \neq M_A$, $a_n < a_n'$ and so $A_{2n} < A_{2n+2}$ and $\bigcup_{n\in\omega} A_{2n} = [m_n, x)$. In a similar way define a sequence of (possibly empty) closed intervals A_{2n+1} in A so that $A_{2n+3} < A_{2n+1}$ and

$\bigcup_{n\in\omega} A_{2n+1} = (x, M_A]$. Now, by $A_n^{(\mu)} \subseteq A^{(\mu)} = \{x\}$, $x \notin A_n$ we have $A_n^{(\mu)} = \varnothing$. Thus A_n is a compact countable ordered set with $\text{ch}(A_n) = (\mu_n, m_n)$, where $\mu_n < \mu$. By the induction hypothesis, $\overline{A_n} \in \mathcal{C}_0$, and so by $A = x\text{-CL}_n A_n$, $\alpha \in \mathcal{C}_0$.

Assume next that $\text{ch}(\alpha) = (\mu, m + 1)$. By Proposition 1.3 let $\alpha = \beta + \tilde{\alpha}$, where $\text{ch}(\beta) = (\mu, m)$ and $\text{ch}(\tilde{\alpha}) = (\mu, 1)$. By the previous case, $\tilde{\alpha} = \text{CL}_n\, \tilde{\alpha}_n$ where $\tilde{\alpha}_n \in \mathcal{C}_0$. By induction, $\beta \in \mathcal{C}_0$. Let $\alpha_0 = \beta$, $\alpha_{2n} = \tilde{\alpha}_{2n+2}$ and $\alpha_{2n+1} = \tilde{\alpha}_{2n+1}$. Then $\alpha_n \in \mathcal{C}_0$ for all n, and $\alpha = \text{CL}_n\, \alpha_n$, whence $\alpha \in \mathcal{C}_0$.

This completes the proof of Theorem 1.2.

2. The Summing Lemma. We say that a set A is *countable* if $|A| \leq \omega$. An ordering $\prec$ of A is called an *$|A|$-ordering* if A ordered by $\prec$ is isomorphic to $|A|$. Let $\prec$ be an $|A|$-ordering of a countable set A, and let $a_0, a_1, \ldots$ be the enumeration of A in the order $\prec$. Define ${}_m[A)_n^{\prec}$ for $m, n \in \omega$ as follows. ${}_m[A)_n^{\prec} = \{a_i\colon m \leq i < n\}$ if $m, n \leq |A|$. ${}_m[A)_n^{\prec} = \varnothing$ if $|A| \leq m$. ${}_m[A)_n^{\prec} = {}_m[A)_{|A|}^{\prec}$ if $|A| \leq n$. For $B \subseteq A$, $m, n \in \omega$ let ${}_m[B)_n^{\prec} = B \cap {}_m[A)_n^{\prec}$. We abbreviate by omitting $\prec$ when no confusion is possible, and write $[B)_n^{\prec}$ for ${}_0[B)_n^{\prec}$, ${}_n[B)^{\prec}$ for ${}_n[B)_{|A|}^{\prec}$.

Let h be a function from the countable set A into a Banach space V. For a finite $B \subseteq A$ let $h(B) = \sum_{a\in B} h(a)$. Let $\prec$ be an $|A|$-ordering of A, and let $B \subseteq A$. Whenever $v = \lim_n h([B)_n)$ exists we write $v = \sum_B^{\prec} h$, and we say that *$\prec$ sums h to v over B*. We say that *$\prec$ sums h over B* whenever $\prec$ sums h over B to some v. If $\mathfrak{B}$ is a family of subsets of A, we say that *$\prec$ sums h over $\mathfrak{B}$* if $\prec$ sums h over B for every $B \in \mathfrak{B}$. We say that *$\prec$ sums h uniformly over $\mathfrak{B}$* if for every $\varepsilon > 0$ there is an $N \in \omega$ such that for every $B \in \mathfrak{B}$ and $k, l \in \omega$ with $N \leq k, l$, we have $\|h({}_k[B)_l)\| < \varepsilon$.

The Summing Lemma. *Let $A = \bigcup_{i\in\omega} A_i$, where $\{A_i\colon i \in \omega\}$ is a disjointed family of countable sets. For each $i \in \omega$ let $\mathfrak{B}_i$ be a family of subsets of A_i, and let $\mathfrak{B} = \{B \subseteq A\colon B \cap A_i \in \mathfrak{B}_i \text{ for all } i \in \omega\}$. Let V be a Banach space and let $h\colon A \to V$. For $i \in \omega$ let c_i be a positive real number and let $\prec_i$ be an $|A_i|$-ordering of A_i so that:*

(2.0) $\sum_{i\in\omega} c_i < \infty$.

(2.1) *$\prec_i$ sums h uniformly over $\mathfrak{B}_i$.*

(2.2) $\|h({}_k[B)_l^{\prec_i})\| < c_i$ *for* $B \in \mathfrak{B}_i$, $k, l \in \omega$.

Then there is an $|A|$-ordering $\prec$ of A such that:

(2.3) $a \prec b$ *iff* $a \prec_i b$ *for* $a, b \in A_i$.

(2.4) *$\prec$ sums h uniformly over $\mathfrak{B}$.*

(2.5) *Let $B \in \mathfrak{B}$ and $B_i = B \cap A_i$. Then*

$$\sum_B^{\prec} h = \sum_{i\in\omega}\left(\sum_{B_i}^{\prec_i} h\right).$$

Proof. Define by induction on $s \in \omega$, m_s, $n_{is} \in \omega$ $(i \in \omega)$ as follows. Let $m_0 = n_{i0} = 0$ $(i \in \omega)$.

Assume that m_{s-1}, $n_{i\,s-1}$ are already defined for $i \in \omega$. By (2.0) we may choose m_s so that $m_{s-1} < m_s$ and

(2.6) $\sum_{m_s < i} c_i < 1/2s$.

By (2.2) we may further choose n_{is} for $i < m_s$ so that $n_{is-1} < n_{is}$ and

$$\left\| h\left({}_k[B)_l^{\prec_i}\right) \right\| < \frac{1}{2 \cdot m_s \cdot s} \quad \text{for } B \in \mathfrak{B}_i,\ n_{is} \leq k, l. \tag{2.7}$$

Let $n_{is} = 0$ for $m_s \leq i$.

Define A_{is} by

$$A_{is} = {}_{n_{is}}[A_i)_{n_{is+1}}^{\prec_i}. \tag{2.8}$$

By construction, $A_i = \bigcup_{s \in \omega} A_{is}$ and $A_{is} \prec_i A_{is+1}$.

Let $\prec$ be any $|A|$-ordering of A satisfying (2.3) and the condition

$$D_s = \bigcup_{\substack{i<s \\ r<s}} A_{ir} \quad \text{is an initial segment of } \prec. \tag{2.9}$$

(For example, $A_{00} \prec A_{01} \prec A_{10} \prec A_{11} \prec A_{02} \prec A_{12} \prec A_{20} \prec A_{21} \prec A_{22} \prec \ldots$, i.e. $A_{ir} \prec A_{\tilde{i}\tilde{r}}$ iff $\max(i, r) < \max(\tilde{i}, \tilde{r})$ or $\max(i, r) = \max(\tilde{i}, \tilde{r})$ and $i < \tilde{i}$ or $\max(i, r) = \max(\tilde{i}, \tilde{r})$, $i = \tilde{i}$ and $r < \tilde{r}$.)

We show next that (2.4) holds. Let $s \in \omega$, $0 < s$. Let $N_s = |D_s|$, let $B \in \mathfrak{B}$, $N_s \leq k, l$ be given, and we demonstrate that $\|h({}_k[B)_l^{\prec})\| < 1/s$. Let $B_i = B \cap A_i$, $B_i' = {}_k[B_i)_l^{\prec}$. By (2.9) $[A)_{N_s}^{\prec} = D_s$ and so, by $N_s \leq k$, $B_i' \cap D_s = \varnothing$. Hence there are $k_i, l_i \in \omega$ such that $n_{is} \leq k_i, l_i$ and $B_i' = {}_{k_i}[B_i)_{l_i}^{\prec_i}$. Thus by (2.7) $\|h(B_i')\| < 1/(2 \cdot m_s \cdot s)$ for $i < m_s$. Also, by (2.2) $\|h(B_i')\| < c_i$. Thus we have by $h({}_k[B)_l^{\prec}) = \sum_{i \in \omega} h(B_i')$ and (2.6)

$$\left\| h\left({}_k[B)_l^{\prec}\right) \right\| \leq \sum_{i<m_s} \|h(B_i')\| + \sum_{m_s \leq i} c_i < m_s \cdot \frac{1}{2 \cdot m_s \cdot s} + \frac{1}{2s} = \frac{1}{s}.$$

Finally, we prove (2.5). Let $B \in \mathfrak{B}$, $B_i = B \cap A_i$ and let $v_i = \sum_{B_i}^{\prec_i} h$. By (2.2) $\|v_i\| \leq c_i$ and so $\sum_{i \in \omega} v_i$ is an absolutely convergent series in V. Thus it is convergent to some $v \in V$ unconditionally, i.e. also under arbitrary rearrangement of its terms. Let $v' = \sum_B^{\prec} h$ and let s be a positive integer. We establish (2.5) by showing $\|v - v'\| < 3/s$.

Let $B_i' = [B_i)_{n_{is}}^{\prec}$, $B_i'' = {}_{n_{is}}[B_i)^{\prec}$. Then $v_i = h(B_i') + \sum_{B_i''}^{\prec_i} h$. By (2.7),

$$\left\| \sum_{B_i''}^{\prec_i} h \right\| \leq \frac{1}{2 \cdot s \cdot m_s} \quad \text{for } i < m_s$$

and so

$$\|v_i - h(B_i')\| \leq \frac{1}{2 \cdot s \cdot m_s}, \qquad i < m_s.$$

Let $D = B \cap D_s$. By (2.9) we have $h(D) = \sum_{i<m_s} h(B_i')$. Hence

$$\left\| \sum_{i<m_s} v_i - h(D) \right\| = \left\| \sum_{i<m_s} (v_i - h(B_i')) \right\| < m_s \cdot \frac{1}{2 \cdot m_s \cdot s} = \frac{1}{2s}.$$

Also, by proof of (2.4)

$$\|v' - h(D)\| < 1/s.$$

By (2.2) and (2.6)

$$\left\| \sum_{m_s<i} v_i \right\| < \sum_{m_s<i} \|v_i\| < \frac{1}{s}.$$

Hence

$$\|v - v'\| = \left\| \left(\sum_{i<m_s} v_i - h(D) \right) - (v' - h(D)) + \sum_{m_s<i} v_i \right\| < \frac{3}{s},$$

and the proof of the Summing Lemma is complete. □

3. Proof of Theorem 1. Let Δ denote the class of compact ordered sets for which Theorem 1 is true, and let $\mathfrak{D} = \{\bar{A}: A \in \Delta\}$. Theorem 1 states $\mathcal{C} \subseteq \mathfrak{D}$.[2] We first list two obvious properties of $\mathfrak{D}$.

PROPOSITION 3.0. (i) $\bar{m} \in \mathfrak{D}$ *for every* $m \in \omega$.
(ii) $\alpha, \beta \in \mathfrak{D} \Rightarrow \alpha + \beta \in \mathfrak{D}$.

Call $f \in C(A, V)$ *representable* (*uniquely representable*) iff there is an enumeration $(J_n)_n$ of f's increments so that $f = \sum_n J_n$ holds in $C(A, V)$ (and whenever $(\bar{J}_n)_n$ is another enumeration of f's increments for which $\sum_n \bar{J}_n$ converges, the identity $f = \sum \bar{J}_n$ holds in $C(A, V)$).

PROPOSITION 3.1. *Let A, B be ordered sets, and let g be a continuous mapping of A onto B satisfying $g(a_1) < g(a_2)$ wherever $a_1 < a_2$. For $f \in C(B, V)$ define $\hat{f} \in C(A, V)$ by $\hat{f} = f \circ g$. Then*:
(i) *f is representable* (*uniquely representable*) *iff $\hat{f}$ is representable* (*uniquely representable*),
(ii) *if $A \in \Delta$ then $B \in \Delta$.*

PROOF. Let $\hat{C}(B, V) = \{\hat{f}: f \in C(B, V)\}$. Clearly, $f \to \hat{f}$ is an isometric isomorphism of $C(B, V)$ onto $\hat{C}(B, V)$. For $b \in B$, $g^{-1}(b)$ is a compact interval of A. Let $a_b = \max g^{-1}(b)$. Let $f \in C(B, V)$, $b \in B$, $v \in V$. Then $J_b^v \in C(B, V)$ is an increment of f iff $J_{a_b}^v = \hat{J}_b^v$ is an increment of $\hat{f}$. Let $(J_n)_n$ be an enumeration of f's increments. Then $(\hat{J}_n)_n$ is an enumeration of $\hat{f}$'s increments, and $\sum_{n\in\omega} J_n$ converges to $h \in C(B, V)$ iff $\sum_{n\in\omega} \hat{J}_n$ converges to $\hat{h} \in \hat{C}(B, V)$. (i) follows, and (ii) follows trivially from (i). □

[2]In fact, $\mathfrak{D}$ is the class of scattered compct order types [M1].

Let A be a countable compact ordered set, and let $f,\ h\colon A \to V$. Let $\mathfrak{B}_A = \{[m_A, a)\colon a \in A\}$. We say that an $|A|$-ordering $\prec$ *sums* h (*uniformly*) *to* f iff $\prec$ sums h (uniformly) over $\mathfrak{B}_A$ and $f(a) = \Sigma^{\prec}_{[m_A, a)} h$ for every $a \in A$.

We prove Theorem 1 rephrased as follows:

THEOREM 1′. *Let A be a countable compact ordered set, V a Banach space, and let $f\colon A \to V$ be a continuous function satisfying $f(m_A) = 0$. Then*: (a′) *There is an $|A|$-ordering $\prec$ of A that sums $f^\dagger$ uniformly to f.*

(b′) *If $\prec$ is any $|A|$-ordering of A that sums $f^\dagger$ uniformly to $\tilde{f}$, then $\tilde{f} = f$.*

PROOF OF (a′). We prove (a′) by induction on $\mathrm{ch}(A) = (\mu, m)$. By Proposition 3.0(i) we may assume $\mu > 0$ and by Proposition 3.0(ii) and Proposition 1.3 it is enough to prove (a′) for $m = 1$. So assume $m = 1$ and let $A^{(\mu)} = \{x\}$.

We may further assume that $m_A < x < M_A$ and x is a core point of A. (*Otherwise*, $x = M_A$, $x < x'$, $x = m_A$ *or* $'x < x$. *Let* $\hat{A} = A \cup \{x_n\colon n \in \omega\}$ *where* $x < x_{n+1} < x_n$ (*and* $x_n < x'$) *if* $x = M_A$ (*if* $x < x'$) *and* $x_n < x_{n+1} < x$ (*and* $'x < x_n$) *if* $x = m_A$ (*if* $'x < x$). *Then* $\mathrm{ch}(\hat{A}) = \mathrm{ch}(A) = (\mu, 1)$, $\hat{A}^{(\mu)} = \{x\}$ *and* x *is a core point of* $\hat{A}$. *Define* $g\colon \hat{A} \to A$ *by* $g(a) = a$, $a \in A$ *and* $g(x_n) = x$. *By Proposition* 3.1(i), (a′) *for* $f \in C(A, V)$ *follows from* (a′) *for* $\hat{f} = f \circ g \in C(\hat{A}, V)$.)

Choose $M_n \in A$ so that the following conditions hold:

(1) $M_{2n} < M_{2n+2} < x < M_{2n+3} < M_{2n+1} \leq M_1 = M_A$ $(n \in \omega)$.

(2) $M_n < M_n'$, $n \neq 1$.

(3) $x = \sup\{M_{2n}\colon n \in \omega\} = \inf\{M_{2n+1}\colon n \in \omega\}$.

(4) $M_{2n} < a, b < M_{2n+3}$ implies $\|f(b) - f(a)\| < 2^{-(2n+3)}$.

This is possible by continuity of f and Corollary 1.5. Let $m_0 = m_A$, $m_{2n+2} = M_{2n}'$ and $m_{2n+1} = M_{2n+3}'$, $n \in \omega$. Define $A_n = [m_n, M_n]$. Then $A = x\text{-}\mathrm{CL}_n A_n$ and A_n is a compact ordered set satisfying $A_n^{(\mu)} = \varnothing$, since $A_n^{(\mu)} \subseteq A^{(\mu)} = \{x\}$ and $x \notin A_n \supseteq A_n^{(\mu)}$. Thus $\mathrm{ch}(A_n) = (\mu_n, m_n)$ where $\mu_n < \mu$.

We may further assume

(5) $f(M_{2n}) = f(m_{2n+2})$, $f(M_{2n+3}) = f(m_{2n+1})$, $n \in \omega$.

(*Otherwise*, *let* $\hat{A}_n = A_n \cup \{c_n\}$ *where* $A_n < c_n$, $c_n \notin A$, *and let* $\hat{A} = x\text{-}\mathrm{CL}_n \hat{A}_n$. *Define* $g\colon \hat{A} \to A$ *by* $g(a) = a$ *for* $a \in A$, $g(c_{2k}) = m_{2k+2}$, $g(c_{2k+3}) = m_{2k+1}$, $g(c_1) = M_1$, *and let* $\hat{f} = f \circ g$. *Let* $\hat{M}_n = M_{\hat{A}_n} = c_n$, $\hat{m}_n = m_{\hat{A}_n} = m_n$. *Then* $\hat{f}(\hat{M}_{2n}) = \hat{f}(\hat{m}_{2n})$, *and* (a′) *for* $\hat{f}$ *implies* (a′) *for* f *by Proposition* 3.1(i).)

Define $f_n \in C(A_n, V)$ by $f_n(a) = f(a) - f(m_n)$, $n \in \omega$. Then $f_n^\dagger = f^\dagger | A_n$ (by (5) this holds true also at M_n). By the induction hypothesis, there is an $|A_n|$-ordering $\prec_n'$ of A_n that sums $f_n^\dagger$ (hence $f^\dagger$) uniformly to f_n. Let $\mathfrak{B}_n = \{[m_n, a)\colon a \in A_n\} \cup \{A_n\}$. Let $K_n \in \omega$ satisfy

$$\left\| f^\dagger\left({}_k[B)_l^{\prec_n'}\right)\right\| < 2^{-n} \quad \text{for } B \in \mathfrak{B}_n,\ K_n \leq k, l. \tag{6}$$

Let $[A_n)_{K_n}^{\prec_n'} = \{a_i\colon 0 \leq i < K_n\}$ where $a_i < a_j$ for $i < j$. Modify the order $\prec_n'$ into an $|A|$-ordering $\prec_n$ by reordering $[A_n)_{K_n}^{\prec_n'}$ in the ordering inherited from

A, that is, let $a_i \prec_n a_j$ for $0 \leqslant i < j < K_n$, and $a \prec_n b$ iff $a \prec'_n b$ for $a \in A_n$, $b \in {}_{K_n}[A_n)^{\prec'_n}$. We shall show that $\prec_n$ satisfies (2.1) and (2.2) (with $h = f^\dagger$ and $c_n > 0$ that satisfy (2.0)).

Now (2.1) is obvious, since $\prec_n$ differs from $\prec'_n$ only on a finite set. We prove (2.2). Let $a_{-1} = m_n$, $a_{K_n} = M_n$. Let $B_j = \{a_i\colon\ 0 \leqslant i < j\}$; and let $a \in (a_{j-1}, a_j]$. Denote $B_a = [m_n, a)$. Then

$$[B_a)_{K_n}^{\prec_n} = B_j \qquad (0 \leqslant j \leqslant K_n). \tag{7}$$

Now, by hypothesis

$$f_n(a_j) = \sum_{B_{a_j}}^{\prec_n} f_n^\dagger = f_n^\dagger(B_j) + \sum_{{}_{K_n}[B_{a_j})^{\prec_n}}^{\prec_n} f_n^\dagger = f^\dagger(B_j) + \sum_{{}_{K_n}[B_{a_j})^{\prec_n}}^{\prec_n} f^\dagger.$$

Now,

$$\sum_{{}_{K_n}[B_{a_j})^{\prec_n}}^{\prec_n} f^\dagger = \lim_l f_n^\dagger\left({}_{K_n}[B_{a_j})_l^{\prec_n}\right), \qquad {}_{K_n}[B_{a_j})_l^{\prec_n} = {}_{K_n}[B_{a_j})_l^{\prec'_n}$$

and so by (6) we have

$$\left\|f^\dagger\left({}_{K_n}[B_{a_j})_l^{\prec_n}\right)\right\| < 2^{-n}. \tag{8}$$

Hence

$$\left\|\sum_{{}_{K_n}[B_{a_j})^{\prec_n}}^{\prec_n} f^\dagger\right\| < 2^{-n}.$$

Also, by (4),

$$\|f_n(a_j)\| = |f(a_j) - f(m_n)| < 2^{-n} \qquad (n > 1).$$

Thus

$$\|f^\dagger(B_j)\| < 2 \cdot 2^{-n} \qquad (n > 1). \tag{9}$$

Now let $a \in (a_{j-1}, a_j]$ $(0 \leqslant j \leqslant K_n)$, and let $k, l \in \omega$. Then

$$f_n^\dagger\left([B_a)_l^{\prec_n}\right) = f_n^\dagger\left([B_a)_{K_n}^{\prec_n}\right) + f_n^\dagger\left({}_{K_n}[B_a)_l^{\prec_n}\right) \qquad (j \leqslant l).$$

Hence, by (7), (8) and (9)

$$\left\|f_n^\dagger\left([B_a)_l^{\prec_n}\right)\right\| < 3 \cdot 2^{-n} \qquad (n > 1).$$

But $f^\dagger({}_k[B_a)_l^{\prec_n}) = f_n^\dagger([B_a)_l^{\prec_n}) - f_n^\dagger([B_a)_k^{\prec_n})$ $(k \leqslant l)$. It follows that

$$\left\|f^\dagger\left({}_k[B_a)_l^{\prec_n}\right)\right\| < 6 \cdot 2^{-n} \qquad (n > 1).$$

Let $c_n = 6 \cdot 2^{-n}$ for $n > 1$. Then c_0, c_1 can be properly chosen so that $\Sigma_n c_n < \infty$ and for every $B \in \mathfrak{B}_n$, $k, l \in \omega$ we have $\|f^\dagger({}_k[B)_l^{\prec_n})\| < c_n$. Thus (2.2) holds.

Let $A' = \bigcup_{n\in\omega} A_n$ and let $\prec'$ be an $|A'|$-ordering of A' so that (2.3)–(2.5) hold.

Let $\prec$ be any $|A|$-ordering of $A = A' \cup \{x\}$ whose restriction to A' is $\prec'$. We show that $\prec$ sums $f^\dagger$ uniformly to f.

Indeed, by (2.4) $\prec$ sums $f^\dagger$ uniformly over $\mathfrak{B} = \{B \subseteq A\colon B \cap A_n \in \mathfrak{B}_n\}$. Now for all $a \in A$, $[m_0, a) \cap A_n \in \mathfrak{B}_n$ and so $\prec$ sums $f^\dagger$ uniformly over $\{[m_0, a)\colon a \in A\}$. We shall show that $\sum^{\prec}_{[m_0, a)} f^\dagger = f(a)$ for each $a \in A$. By (2.3), (5) we have

$$\sum^{\prec}_{A_n} f^\dagger = \sum^{\prec_n}_{[m_n, M_n)} f_n^\dagger = f_n(M_n),$$

so

$$\sum^{\prec_n}_{A_n} f^\dagger = f(M_n) - f(m_n) \tag{10}$$

and for $a \in A_n$, by (2.3)

$$\sum^{\prec}_{[m_n, a)} f^\dagger = f(a) - f(m_n). \tag{11}$$

Now, by (2.5) (10), (11) and by $f^\dagger(x) = 0$ we have for every $a \in A$:

$$\sum^{\prec}_{[m_0, a)} f^\dagger = \sum_{n \in \omega} \left(\sum^{\prec}_{[m_0, a) \cap A_n} f^\dagger \right) = \sum_{A_k < A_n} (f(M_k) - f(m_k)) + f(a) - f(m_n).$$

By $f(m_0) = 0$ we deduce for $a \in A_{2n}$:

$$\sum^{\prec}_{[m_0, a)} f^\dagger = \sum_{k < n} (f(m_{2k+2}) - f(m_{2k})) + f(a) - f(m_{2n}) = f(a).$$

By $f(x) = \lim_n f(m_{2n})$ we have

$$\sum^{\prec}_{[m_0, x)} f^\dagger = \sum_{n \in \omega} (f(m_{2n+2}) - f(m_{2n})) = \lim_n f(m_{2n}) = f(x).$$

Finally, let $a \in A_{2n+1}$. Then

$$\begin{aligned}\sum^{\prec}_{[m_0, a)} f^\dagger &= \sum_{k \in \omega} (f(m_{2k+2}) - f(m_{2k})) \\ &\quad + \sum_{j > n} (f(m_{2j+1}) - f(m_{2j+3})) + f(a) - f(m_{2n+1}) \\ &= f(x) + \sum_{j > n} (f(m_{2j+1}) - f(m_{2j+3})) + f(a) - f(m_{2n+1}).\end{aligned}$$

But $\sum_{n<j<l}(f(m_{2j+1}) - f(m_{2j+3})) = f(m_{2n+1}) - f(m_{2l+1})$. Also, $\lim_l f(m_{2l+1}) = f(x)$, so

$$\sum^{\prec}_{[m_0, a)} f^\dagger = f(x) + (f(m_{2n+1}) - f(x)) + (f(a) - f(m_{2n+1})) = f(a).$$

PROOF OF (b′).[3] We shall rather prove

(b″) Let $\prec'$ be any $|A|$-ordering of A that sums $f^\dagger$ to a continuous function $\tilde{f}$. Then $\tilde{f} = f$.

((b′) *follows, as any* $|A|$-*ordering* $\prec'$ *that sums* $f^\dagger$ *uniformly over* $\{[m_0, a)\colon a \in A\}$ *sums it to a continuous function.*)

(b″) is proved using the Induction Principle (§1). It is obvious for a finite A, and so we have to show that if $A = x\text{-CL}_{n\in\omega} A_n$ and (b″) is true of A_n, $n \in \omega$, it is also true of A. As before, we may assume $A_n \neq \varnothing$ for all n, and setting $m_n = m_{A_n}$, $M_n = M_{A_n}$ we may assume (5).

Let $a \in A_n$ and define $f_n(a) = f(a) - f(m_n)$, $\tilde{f}_n(a) = \tilde{f}(a) - \tilde{f}(m_n)$. Then, $f_n, \tilde{f}_n \in C(A_n, V)$ and $f_n^\dagger = f^\dagger | A_n$. By assumption, $\prec'$ sums $f^\dagger$ over $[m_0, a)$ to $\tilde{f}(a)$ and over $[m_0, m_n)$ to $\tilde{f}(m_n)$, hence $\prec'$ sums $f_n^\dagger$ over $[m_n, a) = [m_0, a) - [m_0, m_n)$ to $\tilde{f}_n(a) = \tilde{f}(a) - \tilde{f}(m_n)$. By the induction hypothesis, $\tilde{f}_n = f_n$. Hence $\tilde{f}(a) = \tilde{f}(m_n) + f_n(a)$ and in particular

$$\tilde{f}(M_n) - \tilde{f}(m_n) = f(M_n) - f(m_n). \tag{12}$$

It is left to show that $\tilde{f}(m_n) = f(m_n)$ and that $\tilde{f}(x) = f(x)$. Now

$$\tilde{f}(m_0) = \sum_{[m_0, m_0)}^{\prec'} f^\dagger = 0 = f(m_0).$$

Assuming $\tilde{f}(m_{2n}) = f(m_{2n})$ we have

$$\tilde{f}(m_{2n+2}) = \sum_{[m_0, m_{2n+2})}^{\prec'} f^\dagger = \sum_{[m_0, m_{2n})}^{\prec'} f^\dagger + \sum_{[m_{2n}, m_{2n+2})}^{\prec} f^\dagger = f(m_{2n}) + \sum_{[m_{2n}, m_{2n+2})}^{\prec'} f^\dagger.$$

By (5), (12):

$$\sum_{[m_{2n}, m_{2n+2})}^{\prec'} f^\dagger = f(m_{2n+2}) - f(m_{2n})$$

hence $\tilde{f}(m_{2n+2}) = f(m_{2n+2})$. Thus $\tilde{f}(m_{2n}) = f(m_{2n})$, $n \in \omega$. Also $\tilde{f}(x) = \lim_n \tilde{f}(m_{2n})$ by continuity of $\tilde{f}$, hence $\tilde{f}(x) = \lim_n f(m_{2n}) = f(x)$. Finally, for $l \in \omega$ we have by (5), (12)

$$\begin{aligned} \tilde{f}(m_{2n+1}) - \tilde{f}(m_{2n+2l+1}) &= \sum_{0<i<l} \left(\tilde{f}(m_{2n+2i+1}) - \tilde{f}(m_{2n+2i+3})\right) \\ &= \sum_{0<i<l} \left(f(m_{2n+2i+1}) - f(m_{2n+2i+3})\right) \\ &= f(m_{2n+1}) - f(m_{2n+2l+1}). \end{aligned}$$

[3](b) is actually a consequence of (a), see [M1]. We use the Induction Principle (§1) to give an independent proof of (b″).

By continuity of $\tilde{f}$, $\lim_l \tilde{f}(m_{2n+2l+1}) = \tilde{f}(x)$. Hence, by $\tilde{f}(x) = f(x)$ and $f(x) = \lim_l f(m_{2n+2l+1})$:

$$\begin{aligned}\tilde{f}(m_{2n+1}) &= \tilde{f}(m_{2n+1}) - \tilde{f}(x) + f(x)\\ &= \lim_l \left(\tilde{f}(m_{2n+1}) - \tilde{f}(m_{2n+2l+1})\right) + f(x)\\ &= \lim_l \left(f(m_{2n+1}) - f(m_{2n+2l+1})\right) + f(x) = f(m_{2n+1}).\end{aligned}$$

This completes the proof of Theorem 1.

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